APPROXIMATION OPERATORS AND TAUBERIAN CONSTANTS

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ABSTRACT

The explicit expression of the smallest constant C satisfying

$$\limsup_{\lambda\to\infty}\left|t^{(1)}_{n(\lambda)}-t^{(2)}_{m(\lambda)}\right|\leqq C.\limsup_{n\to\infty}\left|d_n\right|$$

for all sequences $\{s_n\}$ satisfying $\limsup_{n\to\infty}|d_n|<\infty$, where $\{t^{(1)}_n\}$, $\{t^{(2)}_n\}$ are two generalised Hausdorff transforms of $\{s_n\}$, $\{d_n\}$ is the generalised (C,α)-transform $(0\leqq\alpha\leqq 1)$ of $\{\lambda_n a_n\}$ and $n(\lambda, m(\lambda)$ are suitably related, is obtained. These results are obtained by using new properties of positive approximation operators and generalised Bernstein approximation operators.

1. **Introduction.** Let $\{s_n\}$ $(s_n = a_0 + a_1 + \cdots + a_n,\ n \geqq 0)$ be a real or complex sequence. Denote by $\{t^{(1)}_n\}$ and $\{t^{(2)}_n\}$

$$t^{(j)}_n = \sum_{m=0}^{\infty} a^{(j)}_{nm}\, s_m,\quad n\geqq 0 \quad (j=1,2) \tag{1.1}$$

two linear tranrforms T_1 and T_2 of $\{s_n\}$. Estimates of the form

$$\limsup_{\lambda\to\infty}\left|t^{(1)}_{n(\lambda)}-t^{(2)}_{m(\lambda)}\right|\leqq C.\limsup_{n\to\infty}\left|d_n\right| \tag{1.2}$$

for sequences $\{s_n\}$ satisfying

$$\limsup_{n\to\infty}\left|d_n\right|<\infty \tag{1.3}$$

where $\{d_n\}$ is a certain fixed linear transform of the sequence $\{a_n\}$ $(n \geqq 0)$ and $n(\lambda)\to\infty$, $m(\lambda)\to\infty$ $(\lambda\uparrow\infty)$ depend on the transforms T_1, T_2 and $\{d_n\}$ were considered for the first time by Hadwiger [2]. The smallest value of C satisfying (1.2) for all sequences $\{s_n\}$ satisfying (1.3) is known as the Tauberian constant associated with the pair of transforms T_1, T_2 and $\{d_n\}$.

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In §2 we obtain the explicit expression for the Tauberian constant associated with a pair $\{t_n^{(1)}\}$, $\{t_n^{(2)}\}$ of generalized Hausdorff transforms and a sequence $\{d_n\}$ of generalized Cesàro transforms of order α $(0 \leqq \alpha \leqq 1)$ of the sequence $\{\lambda_n a_n\}$. The cases $\alpha = 0$ and $\alpha = 1$ were considered before for various classes of pairs of transforms. The case $0 < \alpha < 1$ was considered for only one pair of transforms before (see [7]).

In §3 we obtain general results about properties of pairs of approximation operators which extend results of [8].

In §4 we give various properties of generalized Bernstein approximation operators.

The results of §§3 and §4, which are interesting by themselves too, are used in §5 in the proof of the results of §2.

§2. **Tauberian constants.** For a fixed sequence $\{b_n\}$ $(n \geqq 0)$ denote by $b_n!$ the product $b_0 \cdots b_n$. For $n \geqq m \geqq 0$ denote by $d_n!/d_m!$ the product $d_{m+1} \cdots d_n$ if $m < n$ and the number 1 if $m = n$, even if some of the numbers b_k are zero or ∞. The symbol $b_n!$ *does not* denote here $\Gamma(b_n + 1)$. The Binomial coefficien $\binom{\alpha}{\beta}$ is defined by

$$\binom{\alpha}{\beta} = \Gamma(\alpha+1)/\{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)\}.$$

For a fixed sequence $\{\lambda_k\}$ $(k \geqq 0)$ satisfying

$$(2.1) \qquad 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \uparrow \infty, \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k} = +\infty$$

the regular generalized Hausdorff transformation $\{H_n(\beta)\}$ $(n \geqq 0)$ of a sequence $\{s_n\}$ is defined (see [6]) by

$$H_n(\beta) = \int_0^1 \sum_{k=0}^{n} P_{nk}(t) s_k d\beta(t), \qquad n = 0,1,2,\cdots$$

where $\beta(t)$ is a function satisfying

$$(2.2) \qquad \beta(t) \text{ is normalized and bounded variation in } [0,1],$$
$$\beta(0) = \beta(0+) = 0 \text{ and } \beta(1) = 1$$

and (see, [11, §2.7, (8)]).

$$(2.3) \qquad P_{nk}(t) = \begin{cases} (-1)^{n-k}\{\lambda_n!/\lambda_k!\}[t^{\lambda_k}, \cdots, t^{\lambda}] & \text{if } 0 \leqq k \leqq n \\ 0 & \text{if } k > n \end{cases}$$

where the divided differences of a sequence $\{\mu_n\}$ $(n \geqq 0)$ with respect to the fixed sequence $\{\lambda_k\}$ is defined by

(2.4) $[\mu_k] = \mu_k$, $[\mu_k, \cdots, \mu_{n+1}] = \{[\mu_k, \cdots, \mu_n] - [\mu_{k+1}, \cdots, \cdots \mu_{n+1}]\}/(\lambda_k - \lambda_{+1})$.

With the fixed sequence $\{\lambda_n\}$ we associate the sequence $\rho_n = \lambda_n/\lambda_1$ $(n \geqq 0)$, and we define for n, $k \geqq 0$

$$\alpha_{nk} = [(1 - 1/\rho_n)!/(1 - 1/\rho_k)!]^{1/\lambda_1}. \tag{2.5}$$

We have

$$\begin{cases} \alpha_{nk}/\alpha_{mk} = \alpha_{nm} & \text{for } n, m, k = 1, 2, \cdots \\ 0 \leqq \alpha_{nk} \leqq 1 & \text{for } 0 \leqq k \leqq n \\ \alpha_{n0} = 0 & \text{for } n \geqq 1 \\ \alpha_{nk} < \alpha_{n,k+1} & \text{for } n = 1, 2, \cdots \text{ and } k = 0, 1, 2 \cdots \end{cases} \tag{2.6}$$

For a sequence $\{s_n\}$ $(s_n = a_0 + \cdots + a_n)$ define for a fixed $\alpha \geqq 0$ the sequence $\{a_n^{(\alpha)}\}$ $(n \geqq 0)$ as the Generalized Hausdorff tranrform of the sequence $\{\lambda_n a_n\}$ corresponding to the function $\Psi_\alpha(t) = 1 - (1-t)^\alpha$ if $\alpha > 0$ and $\Psi_0(t) = 0$ $(0 \leqq t < 1)$, $\Psi_0(1) = 1$. That is, $\{a_n^{(\alpha)}\}$ is the generalized Cesàro transform of the sequence $\{\lambda_n a_n\}$ $(n \geqq 0)$.

THEOREM 2.1: *Let (H, β) and (H, γ) be regular generalized Hausdorff transformations such that* $\int_0^1 \frac{|\beta(t)|}{t} dt < \infty$ *and* $\int_0^1 \frac{|\gamma(t)|}{t} dt < \infty$. *Let α be a fixed number satisfying $0 \leqq \alpha < 1$. For a sequence $\{s_n\}$ satisfying $a_n^{(\alpha)} = 0(1)$ we have, for each q, $0 < q \leqq 1$, and any pair of integral valued functions $n(\lambda)$, $m(\lambda)$ satisfying $n(\lambda) \to \infty$ and $m(\lambda) \to \infty$, $\alpha_{n(\lambda), m(\lambda)} \to q$ as $\lambda \to \infty$, and ultimately $n(\lambda) \geqq m(\lambda)$,*

$$\limsup_{\lambda \to \infty} |H_n(\beta) - H_m(\gamma)| \leqq F_q^{(\alpha)} \cdot \limsup_{n \to \infty} |a_n^{(\alpha)}| \tag{2.7}$$

where

$$\Gamma(1-\alpha)\Gamma(1+\alpha)F_q^{(\alpha)} = \int_0^q \left| d_x \left\{ \int_x^1 t^\alpha d_t \left(\int_t^1 (u-t)^{-\alpha} \frac{\beta(u)}{u} du \right) + \right.\right.$$
$$+ \int_0^x t^\alpha d_t \left(\int_q^1 \frac{(u-t)^{-\alpha}}{u} du \right) - \int_{x/q}^1 t^\alpha d_t \left(\int_t^1 (u-t)^{-\alpha} \frac{\gamma(u)}{u} du \right\} \right| +$$
$$+ \int_q^1 x^\alpha \left| d_x \left(\int_x^1 (t-x)^{-\alpha} \frac{1-\beta(t)}{t} dt \right| . \tag{2.8}$$

The constant $F_q^{(\alpha)}$ *is the best in the following sense. There is a real sequence* $\{s_n\}$ *satisfying* $a_n^{(\alpha)} = O(1)$ *and such that both members of inequality*(2.7) *are equal.*

Theorem (2.1) for $\alpha = 0$ and either $\beta(t)$ or $\gamma(t)$ equal to the function which is zero for $0 \leqq t < 1$ and is equal to 1 for $t = 1$ is Theorem (3.1) of [8]. Theorem (2.1) for $\alpha = 0$ and the additional assumptions that $\int_0^1 x^{-1}(\int_0^x |d\beta(t)|)dx < \infty$ and $\int_0^1 x^{-1}(\int_0^x |d\gamma(t)|)dx < \infty$ was proved by A. Meir in [12].

THEOREM 2.2. *Let* (H,β) *and* (H,γ) *be regular generalized Hausdorff transformations. For a sequence* $\{s_n\}$ *satisfying* $a_n^{(1)} = 0(1)$, *each* q, $0 < q \leqq 1$, *and any pair of integral valued functions* $n(\lambda)$, $m(\lambda)$ *satisfying the same assumptions as in Theorem* (2.1) *we have the following results*: (I) *If* $0 < q < 1$ *or* $q = 1$ *and* $m(\lambda) < n(\lambda)$ *for* $\lambda > \Lambda$,

$$\int_0^t \frac{|\beta(t)|}{t}dt < \infty, \quad \int_0^1 \frac{|\gamma(t)|}{t}dt < \infty$$

and either a) $\gamma(t)$ *is absolutely continuous in each interval* $[\delta, 1-\delta]$ $(0 < \delta < \frac{1}{2})$ *and* $\beta(t)$ *is continuous at* $t = q$, *or* b) $\beta(t)$ *is absolutely continuous in each interval* $[\delta, q-\delta]$ $(0 < \delta < \frac{1}{2}q)$, $\beta(t)$ *is continuous in some interval* $[0, q+\varepsilon]$ $(\varepsilon > 0)$ *and* $\gamma(t)$ *is continuous in* $0 \leqq t < 1$, *then*

$$\limsup_{\lambda\to\infty} |H_n(\beta) - H_m(\gamma)| \leqq F_q^{(1)} \limsup_{n\to\infty} |a_n^{(1)}| \tag{2.9}$$

where

$$F_q^{(1)} = \int_0^q u \left| d\left(\frac{\beta(u)-\gamma_1(u/q)}{u}\right)\right| + |1-\gamma(1-0)| + \int_q^1 u \left| d\left(\frac{1-\beta(u)}{u}\right)\right| \tag{2.10}$$

($\gamma_1(t) = \gamma(t)$ $(0 \leqq t < 1)$, $\gamma_1(1) = \gamma(1-0)$ and for $q = 1$

$$\int_0^1 u \left| d\left(\frac{1-\beta(u)}{u}\right)\right| \equiv |1-\beta(1-0)|).$$

(II) *If* $q = 1$, $m(\lambda) = n(\lambda)$ *for* $\lambda > \Lambda$ *and*

$$\int_0^1 \frac{|\beta(t)-\gamma(t)|}{t}dt < \infty,$$

then (3.9) *is true with*

$$F_q^{(1)} = \int_0^1 u \left| d\,\frac{\beta_1(u)-\gamma_1(u)}{u}\right| + |\beta(1-0)-\gamma(1-0)| \tag{2.11}$$

($\beta_1(t)$ is defined in the same way as $\gamma_1(t)$. In both cases (I) *and* (II) *the constant $F_q^{(1)}$ is the best in the following sense. There are real sequences $\{s_n\}$ such that $a_n^{(1)} = 0(1)$ and both sides of* (2.9) *are equal.*

Theorem (2.2) with the assumptions that $\gamma(t)$ and $\beta(t)$ are continuous in $0 \leqq t \leqq 1$ and have continuous derivatives for $0 < t < 1$ was proved by A. Meir [13].

Theorem (2.2) with $\gamma(t)$ equal to the function which is zero for $0 \leqq t < 1$ and is equal to 1 for $t = 1$ is Theorem (3.2) of [8]. The constant B_q which appears there should be corrected to $F_q^{(1)}$ as in (2.11) here.

Results similar to Theorems (2.1) and (2.2) can be obtained for the difference of two regular $[H, \alpha, \beta(t)]_n$ transforms (see [8, p. 191]).

The proof of Theorems (2.1) and (2.2) is given in §5.

§3. Some results about positive approximation operators.

In this paper each function $g(t)$ of bounded variation in a closed interval $[a, b]$ is extended to R^1 by the definition $g(t) = g(a)$ for $t < a$ and $g(t) = g(b)$ for $t > b$.

Unless otherwise stated we assume that each function $g(t)$ of bounded variation in a closed interval $[a, b]$ is normalized that is $g(a) = 0$,

$$g(t) = \tfrac{1}{2}[g(t-0) + g(t+0)] \text{ for } a < t < b.$$

We use here the definition of Riemann-Stieltjes integrals given in [4]. We use freely the fact that the existence of a Riemann-Stieltjes integral $\int_0^1 f(t)dg(t)$ where $g(t)$ is of a bounded variation in $[0.1]$ implies the existence of the Lebesgue-Stieltjes integral $\int_{[0.1]} f(t)d\mu_g$ in sense given in [14] where μ_g is the Lebesgue-Stieltjes measure in R^1 generated by the function $g(t)$ extended to R^1 as mentioned above and modified so as to be continuous on the right in R^1.

We prove here the following result.

THEOREM 3.1. *Let $s_n(u)$ and $h_x(u)$ $(n \geqq 0,\ x \geqq 0)$ be normalized functions of bounded variation in $[0,1]$ and $[0, a]$ $(0 < a \leqq +\infty)$, respectively, continuous at $u = 0$ and satisfying*

$$(3.1) \quad \int_0^1 \left| d\{s_n(u) - s_0(u)\} \right| \to 0 \ (n \to \infty), \quad \int_0^a \left| d\{h_x(u) - h_0(u)\} \right| \to \quad (x \to \infty).$$

For $n, k = 0, 1, 2, \cdots$ and $x \geqq 0$ let $P_{nk}(u)$ and $P^*_{xk}(u)$ be non-negative continuous functions in $[0,1]$ and $[0, a]$, respectively, satisfying

$$(3.2)\qquad \sup_{\substack{n\geqq 0\\ 0\leqq u\leqq 1}} \sum_{k=0}^{\infty} P_{nk}(u) \equiv K < \infty, \quad \sup_{\substack{x\geqq 0\\ 0\leqq u\leqq a}} \sum_{k=0}^{\infty} P^*_{xk}(u) \equiv N < \infty.$$

For $n = N, N+1, \cdots, k = 0,1,2,\cdots$ and $x \geqq x_0 \geqq 0$ let α_{nk} and ρ_{xk} be non-negative real numbers and functions (of x) satisfying

$$(3.3)\qquad \begin{cases} 0 \leqq \alpha_{nk} \leqq 1 & \text{if } P_{nk}(u) \not\equiv 0 \quad \text{for } 0 \leqq u \leqq 1, \\ \alpha_{n0} = 0 & \text{for } n \geqq N, \\ \alpha_{nk} = \alpha_{nq} & \text{for } n \geqq N \quad \text{if, and only if, } k = q, \\ 0 \leqq \rho_{xk} \leqq a & \text{if } P^*_{xk}(u) \not\equiv 0 \quad \text{for } 0 \leqq u \leqq a, \\ \rho_{x0} = 0 & \text{for } x \geqq X \text{ (all sufficiently large } x) \\ \rho_{xk} = \rho_{xq} & \text{for } x \geqq X \quad \text{if, and only if, } k = q. \end{cases}$$

Suppose for each real and bounded function $f(t)$ in $[0,1]$ and each real and bounded function $g(t)$ in $[0,a]$ we have at each point of continuity $t = u\,(0 < u \leqq 1$ of $f(t)$ and $t = u$ $(0 < u \leqq a)$ of $g(t)$

$$(3.4)\qquad \begin{cases} \lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} P_{nk}(u) f(\alpha_{nk}) = f(u) \\ \lim\limits_{x\to\infty} \sum\limits_{k=0}^{\infty} P^*_{xk}(u)\, g(\rho_{xk}) = g(u) \end{cases}$$

Let $n(\lambda)$ $(\lambda > 0)$ be a positive integral valued function $x(\lambda)$ $(\lambda > 0)$ a real positive function satisfying

$$(3.5)\qquad \begin{cases} n(\lambda) \to \infty, \quad x(\lambda) \to \infty \quad \text{as } \lambda \to +\infty \\ \text{For } \rho_{x(\lambda),n(\lambda)} \equiv q(\lambda) \text{ we have } 0 < q(\lambda) \leqq a \text{ if } \lambda > \Lambda \\ q(\lambda) \to q \qquad (\lambda \to +\infty \qquad \text{and} \quad 0 < q \leqq a \end{cases}$$

First conclusion: *For a fixed value of t $0 \leqq t \leqq a$ we have for $n \equiv n(\lambda), x \equiv x(\lambda)$*

$$(3.6)\qquad \lim_{\lambda\to\infty} \sum_{\substack{k\\ \rho_{xk}\leqq t}} \left| \int_0^1 P_{nk}(u) ds_n(u) - \int_0^a P^*_{xk}(u) dh_x(u) \right| = \int_0^t |dS(u)|$$

where $S(u) \equiv s_0(u/q) - h_0(u)$, if one of the following two assumptions is satisfied. (I) *The function $S(u)$ is continuous at $u = t$ if $0 \leqq t < a$ (if $t = a$ it is not assumed that $S(u)$ is continuous at $u = t = a$) and for $x \equiv x(\lambda)$, $n \equiv n(\lambda)$*

$$(3.7)\qquad \lim_{\lambda\to\infty} \sum_{\substack{k\\ \rho_{xk}\leqq t}} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} ds_0(u) \right| = 0.$$

(II) *The function* $S(u)$ *is continuous at each point of the interval* $[0,\min(t,q)] - \{\delta_{tq}\cdot\delta_{qa}\}$ *(where* δ_{tq} *is the Kronecker symbol), we have* $\rho_{xk}/\alpha_{nk} = \rho_{xn}$ *for any* $x,n,k > 0$ *and in case* $t \geqq q$ *the function* $h_0(t)$ *is continuous at each point of the interval* $[q,t] - \{1\}$ *and for* $n \equiv n(\lambda)$, $x \equiv x(\lambda)$

$$\lim_{x\to\infty} \sum_{\substack{k \\ \rho_{xk}\leqq t}} \left| \int_0^1 \{P_{nk}(u) - P^*_{xk}(qu)\} dh_0(qu) \right| = 0. \tag{3.8}$$

Second conclusion: *We have for* $n \equiv n(\lambda)$, $x \equiv x(\lambda)$

$$\lim_{x\to\infty} \sum_{\substack{k \\ \rho_{xk}\leqq q(\lambda)}} \left| \int_0^1 P_{nk}(u)ds_n(u) - \int_0^a P^*_{xk}(u)dh_x(u) \right| = \int_0^q |dS(u)| \tag{3.9}$$

if one of the following two assumptions is satisfied. (III) *The function* $S(u)$ *is continuous at* $t = q$ *and condition* (3.7) *is satisfied for some* t *such that* $q < t \leqq a$ *if* $q < a$ *and* $t = a$ *if* $q = a$. (IV) *The function* $S(u)$ *is continuous at each point of* $0 \leqq u \leqq q$, *for some* $\varepsilon > 0$ *the function* $h_0(t)$ *is continuous at each point* $q \leqq u \leqq q + \varepsilon$ *and for some* t *such that* $q \leqq t \leqq a$ *if* $q < a$ *and* $t = a$ *if* $q = a$.

Theorem (3.1) remains true if the interval $[0,1]$ or $[0,a]$ or both are replaced by finite or infinite intervals, and $n(\lambda)$, $x(\lambda)$ are real functions or integral valued functions, if the necessary obvious changes are made in the assumptions of the theorem. Also, it is possible to obtain results similar to those of [8, §2].

Theorem (3.1) for $h_x(u) \equiv 0$ and some additional restrictions on the functions $s_n(u)$ was proved in [8, §2].

In the proof of Theorem (3.1) we use the following two results.

LEMMA 3.1. *Let* $\alpha(t)$ *be a normalized function of bounded variation in* $[a,b]$. *Let* $\alpha_n(t)$ $(n \geqq 1)$ *be functions of bounded variation in* $[a,b]$ *not necessarily normalized. Suppose*

$$\lim_{n\to\infty} \alpha_n(a) = \alpha(a), \qquad \lim_{n\to\infty} \alpha_n(b) = \alpha(b)$$

and

$$\lim_{n\to\infty} \alpha_n(u) = \alpha(u) \qquad \text{at each point of continuity of } \alpha(u).$$

Then

$$\int_a^b |d\alpha(u)| \leqq \liminf_{n\to\infty} \int_a^b |d\alpha_n(u)|.$$

Lemma (3.1) is an extension of [16, Corollary 16.4, p. 32]. Its proof is similar.

LEMMA 3.2. *Suppose the functions $P_{nk}(u)$ and the numbers $\alpha_{nk}\,(n,k = 0,1,2,$ satisfy the assumptions of Theorem* (3.1). *Let $\beta(u)$ be a function of bounded variation in $[0,1]$ and continuous for $u = 0$. For a fixed t, $0 \leqq t \leqq 1$ we have*

$$\lim_{n\to\infty} \sum_{\substack{k \\ \alpha_{nk}\leqq t}} \int_0^1 P_{nk}(u)d\beta(u) = \lim_{n\to\infty} \int_0^1 \Big\{ \sum_{\substack{k \\ \alpha_{nk}\leqq t}} P_{nk}(u)\ d\beta(u) = \int_0^t d\beta(u)$$

$$\lim_{n\to\infty} \sum_{\substack{k \\ \alpha_{nk}\geqq t}} \int_0^1 P_{nk}(u)d\beta(u) = \lim_{n\to\infty} \int_0^1 \Big\{ \sum_{\substack{k \\ \alpha_{nk}\geqq t}} P_{nk}(u)\ d\beta(u) = \int_t^1 d\beta(u)$$

if for $0 \leqq t < 1$ $\beta(u)$ is continuous at $u = t$.

Proof. The equality

$$\sum_{\substack{k \\ \alpha_{nk}\leqq t}} \int_0^1 P_{nk}(u)d\beta(u) = \int_0^1 \sum_{\substack{k \\ \alpha_{nk}\leqq t}} P_{nk}(u)\ d\beta(u)$$

follows by Beppo-Levi's theorem if we note that a Riemann-Stieltjes integral is equal to the corresponding Lebesgue-Stieltjes integral. Also by (3.4) (for the function $f(u) = 1$ $(0 \leqq u \leqq t)$, $f(u) = 0$ $(t < u \leqq 1)$ if $0 \leqq t < 1$ or $f(u) = 1$ $(0 \leqq u \leqq 1)$ if $t = 1$) we get

$$\lim_{n\to\infty} \sum_{\substack{k \\ \alpha_{nk}\leqq t}} P_{nk}(u) = \lim_{n\to\infty} \sum_{k=0}^{\infty} P_{nk}(u)f(\alpha_{nk}) = f(u),$$

for $0 < u \leqq 1$, $u \neq t$ if $0 \leqq t < 1$ and for $0 < u \leqq 1$ if $t = 1$. Since the function $\beta(u)$ is continuous at $u = 0$ and $u = t$ (if $0 < t \leqq 1$) we see that we have

$$\lim_{n\to\infty} \sum_{\substack{k \\ \alpha_{nk}\leqq t}} P_{nk}(u) = f(u)$$

almost everywhere in $[0,1]$ with respect to the Lebesgue-Stieltjes measure generated by $\beta(u)$. Hence by Lebesgue's dominated convergence theorem we have

$$\lim_{n\to\infty} \sum_{\substack{k \\ \alpha_{nk}\leqq t}} \int_0^1 P_{nk}(u)d\beta(u) = \int_0^1 f(u)d\beta(u) = \int_0^1 d\beta(u).$$

The proof of the second consequence of the lemma is the same.

Proof of the first conclusion of Theorem (3.1) for assumption (II). Define for $x \equiv x(\lambda)$, $n \equiv n(\lambda)$

$$(3.10)\quad \alpha_\lambda(p) = 0,\ \alpha_\lambda(v) = \sum_{\substack{k \\ \rho_{nk}\leqq v}} \Big\{ \int_0^1 P_{nk}(u)ds_n(u) - \int_0^a P^*_{xk}(u)dh_x(u) \Big\}, 0 < v \leqq a.$$

We have for $0 < v \leqq t$

$$\sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \left\{ \int_0^1 P_{nk}(u) ds_n(u) - \int_0^a P^*_{xk}(u) dh_x(u) \right\} = \sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \int_0^1 P_{nk}(u) d\{s_0(u) - h_0(qu)\}$$

$$- \sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \int_q^a P^*_{xk}(u) dh_0(u)$$

$$+ \sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \int_0^1 \{P_{xk}(u) - P^*_{xk}(qu)\} dh_0(qu)$$

$$(3.11) \qquad + \sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \int_0^1 P_{nk}(u) d\{s_n(u) - s_0(u)\}$$

$$+ \sum_{\substack{k \\ \rho_{\lambda k} \leqq v}} \int_0^a P^*_{xk}(u) d\{h_0(u) - h_x(u)\}$$

$$\equiv J_\lambda^{(1)} + J_\lambda^{(2)} + J_\lambda^{(3)} + J_\lambda^{(4)} + J_\lambda^{(5)}.$$

By (3.1) and (3.2) we get

$$(3.12) \qquad \lim_{\lambda \to \infty} J_\lambda^{(4)} = \lim_{\lambda \to \infty} J_\lambda^{(5)} = 0.$$

By (3.8) we get

$$(3.13) \qquad \lim_{\lambda \to \infty} J_\lambda^{(3)} = 0.$$

By the assumptions on the continuity of the function $h_0(u)$ and Lemma (3.2) we get for $0 < t \leqq x$

$$(3.14) \qquad \lim_{x \to \infty} J_\lambda^{(2)} = \begin{cases} 0 & \text{if } v < q \\ -\int_q^v dh_0(u) & \text{if } q \leqq v \leqq a. \end{cases}$$

By (3.3) and (3.5) we have for $\lambda > \Lambda_2$

$$(3.15) \qquad \{k \,|\, \rho_{xk} \leqq v\} = \{k \,|\, \alpha_{nk} \leqq v/q(\lambda)\}.$$

By Lemma (3.2) and (3.15) we get

$$(3.16) \qquad \lim_{\lambda \to \infty} J_\lambda^{(1)} = \int_0^1 dS(u) \qquad \text{if } q < v \leqq t.$$

Suppose $\lambda > \Lambda_3$. We have by (3.15)

$$J_\lambda^{(1)} = \left\{ \sum_{\substack{k \\ \alpha_{nk} \leqq v/q}} \pm \sum_{\substack{k \\ \alpha_{nk} \varepsilon (v/q(\lambda), v/q]}} \right\} \int_0^1 P_{nk}(u) dS(qu) \tag{3.17}$$

$$\equiv J_\lambda^{(11)} \pm J_\lambda^{(12)} \quad .$$

Assume $0 < v < q$, $0 < v \leqq t$. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $0 < v/q - \varepsilon_1 < v/[q(\lambda)] < v/q + \varepsilon_2 \leqq 1$ and such that the points $v/q - \varepsilon_1$, $v/q + \varepsilon_2$ are points of continuity of $S(qu)$. Then by Lemma (3.2) we get

$$\limsup_{\lambda \to \infty} J_\lambda^{(12)} \big| \leqq \int_{v/q - \varepsilon_1}^{v/q + \varepsilon_2} \big| dS(qu) \big| \to 0 \quad (\varepsilon_1, \varepsilon_2 \downarrow 0) \tag{3.18}$$

since $S(qu)$ is continuous at $u = v/q$. By Lemma (3.2), (3.17), (3.18) and the continuity of $S(qu)$ at $u = v/q$ we get

$$\lim_{\lambda \to \infty} J_\lambda^{(1)} = \int_0^{v/q} dS(qu) = \int_0^v dS(u) \qquad \text{if } 0 < v < q, \; 0 < v \leqq t. \tag{3.19}$$

Similarly for $0 < t \leqq x$, $t = q < 1$ we get by Lemma (3.2), since $S(qu)$ is continuous at $u = 1 = (t/q)$

$$\lim_{\lambda \to \infty} J_\lambda^{(1)} = \int_0^v dS(u) \qquad \text{if } 0 < v = q < a, \; 0 < v \leqq t. \tag{3.20}$$

If $0 < t \leqq x$, $t = q = 1$, then by Lenmma (3.2) we get

$$\lim_{\lambda \to \infty} J_\lambda^{(1)} = \int_0^v dS(u) \qquad \text{if } 0 < v = q = a, \; 0 < v \leqq t. \tag{3.21}$$

Now by (3.11), (3.12), (3.13), (3.14), (3.16), (3.19), (3.20) and (3.21) we have for $0 \leqq v \leqq t$

$$\lim_{\lambda \to \infty} \alpha_x(v) = \int_0^v dS(u). \tag{3.22}$$

By Lemmas (3.1) and (3.2), (3.1), (3.8), (3.11), and (3.15) we get

$$\begin{aligned} \int_0^t \big| dS(u) \big| &\leqq \liminf_{\lambda \to \infty} \int_0^t \big| d\alpha_\lambda(u) \big| \\ &\leqq \limsup_{\lambda \to \infty} \int_0^t \big| d\alpha_\lambda(u) \big| \\ &= \limsup_{\lambda \to \infty} \sum_{\substack{k \\ \rho_{xk} \leqq t}} \left| \int_0^1 P_{nk}(u) ds_n(u) - \int_0^a P^*_{xk}(u) dh_x(u) \right| \end{aligned}$$

$$\leqq \limsup_{\lambda\to\infty} \sum_{\substack{k \\ \rho_{xk}\leqq t}} \left| \int_0^1 P_{nk}(u)d\{s_0(u)-h_0(qu)\}\right|$$

$$+\left|\int_q^a P^*_{xk}(u)dh_0(u)\right|$$

$$+\left|\int_0^1 \{P_{nk}(u)-P^*_{xk}(qu)\}dh_0(qu)\right|$$

$$+\left|\int_0^1 P_{nk}(u)d\{s_n(u)-s_0(u)\}\right|$$

$$+\left|\int_0^a P^*_{xk}(u)d\{h_0(u)-h_x(u)\}\right|\Bigg\}$$

$$\leqq \limsup_{\lambda\to\infty}\int_0^1 \sum_{\substack{k\\ \rho_{xk}<t}} P_{nk}(u)\ \left|d\{s_0(u)-h_0(qu)\}\right|$$

$$+\limsup_{\lambda\to\infty}\int_q^a\Bigg\{\sum_{\substack{k\\ \rho_{xk}<t}} P^*_{xk}(u)\Bigg\}\left|dh_0(u)\right|$$

(by Lemma (3.1), (3.15) and the continuity of $S(u)$)

$$=\int_0^t \left|dS(u)\right|.$$

This completes the proof.

Proof of the first conclusion of Theorem (3.1) for assumption (I).

The proof is similar to that for assumption (II) but instead of (3.11) we use now for $0<v<t$.

$$\sum_{\substack{k\\ \rho\leqq k}}\Bigg\{\int_0^1 P_{nk}(u)ds_n(u)-\int_0^a P^*_{xk}(u)dh_x(u) = \sum_{\substack{k\\ \rho_{xk}\leqq v}}\int_0^a P^*_{xk}(u)\,dS(u)$$

$$+\sum_{\substack{k\\ \rho_{xk}\leqq v}}\int_0^1 P_{nk}(u)d\{s_n(u)-s_0(u)\}$$

(3.23)

$$+\sum_{\substack{k\\ \rho_{xk}\leqq v}}\int_0^a P^*_{xk}(u)d\{h_0(u)-h_x(u)\}$$

$$+\sum_{\substack{k\\ \rho_{xk}\leqq v}}\int_0^1\{P_{nk}(u)-P^*_{xk}(qu)\}ds_0(u).$$

Q.E.D.

Proof of the second conclusion of Theorem (3.1) for assumption (III).

We have for $0 < q < a$ $(0 < q - \varepsilon_1 < q(\lambda) < q + \varepsilon_2 < 1)$

$$J_\lambda^{(1)} \equiv \sum_{\substack{k \\ \rho_{nk} \leqq q - \varepsilon_1}} \left| \int_0^1 P_{nk}(u) ds_n(u) - \int_0^1 P_{xk}^*(u) dh_x(u) \right|$$

$$\leqq \sum_{\substack{k \\ \rho_{nk} \leqq q(\lambda)}} \left| \cdots \right| \leqq \sum_{\substack{k \\ \rho_{nk} \leqq q + \varepsilon_2}} \left| \cdots \right| \equiv J_\lambda^{(2)}.$$

By the first conclusion of Theorem (3.1) for assumption (I) we get

$$\lim_{\lambda \to \infty} J_\lambda^{(1)} = \int_0^{q-\varepsilon_1} \left| dS(u) \right|, \qquad \lim_{\lambda \to \infty} J_\lambda^{(2)} = \int_0^{q+\varepsilon_2} \left| dS(u) \right|$$

if $\varepsilon_1, \varepsilon_2 > 0$ are such that $S(u)$ is continuous at $u = q - \varepsilon_1$, $u = q + \varepsilon_2$. As $S(u)$ is continuous at $u = q$ we get

$$\lim_{\varepsilon_1 \downarrow 0} \lim_{\lambda \to \infty} J_n^{(1)} = \lim_{\varepsilon_2 \downarrow 0} \lim_{\lambda \to \infty} J_n^{(2)} = \int_0^q \left| dS(u) \right|.$$

This proves our theorem for $0 < q < 1$. For $q = 1$ the proof is similar. Q.E.D.

Proof of the second conclusion of Theorem (3.1) for assumption (IV).

The proof is similar to the proof for assumption (III) but now we use the first conclusion of the theorem for assumption (II). Q.E.D.

THEOREM 3.2. *Suppose* $P_{xk}^*(u)$ and ρ_{xk} satisfy the assumption of Theorem (3.1). *If $g(u)$ is Riemann-integrable in* $[0, b]$ *where* $0 < b \leqq a$, $b < +\infty$, *then*

$$\lim_{x \to \infty} \sum_{k=0}^{\infty} \left| \int_0^b P_{xk}^*(u) g(u) du - g(\rho_{xk}) \int_0^b P_{xk}^*(u) du \right| = 0.$$

Proof. As $g(u)$ is Riemann-integrable in $[0, b]$ it is continuous a.e. in $[0, b]$. Hence the function $h_u(t) = g(u) - g(t)$ is bounded and continuous at $t = u$ for almost all values of u in $[0, b]$. Hence by Lebesgue's bounded convergence theorem, (3.2) and (3.4) applied to the function $h_u(t)$ we get

$$\left| \sum_{k=0}^{\infty} \int_0^b P_{xk}^*(u) g(u) du - g(\rho_{xk}) \int_0^b P_{xk}^*(u) du \right| \leqq \int_0^b \left\{ \sum_{k=0}^{\infty} P_{xk}^*(u) h_u(\rho_{xk}) \right\} du \to 0$$

$$(x \to \infty)$$

Q.E.D.

Theorem (3.2) for $p_{xk}^*(u) = \binom{[x]}{k} u^k (1-u)^{[x]-k}$ and $g(u)$ continuous was essentially proved by A. Meir [13, Lemma 2].

4. Some properties of the Generalized Bernstein Polynomials.

For a fixed sequence $\{\lambda_k\}$ $(k \geqq 0)$ satisfying (2.1) the functions $p_{nk}(t)$ and the numbers α_{nk} are, in this section, those defined by (2.3) and (2.5).

The following result was proved by Hirshmann and Widder (see [11, p. 4 Theorem 2.8.2]).

THEOREM 4.I. *If $f(u)$ is a real and bounded function in $[0,1]$ continuous at the point $u = x$ $(0 \leqq x \leqq 1)$, then*

$$(4.1) \qquad \lim_{n\to\infty} \sum_{k=0}^{n} p_{nk}(x) f(\alpha_{nk}) = f(x).$$

It is known (see [11, p. 46, (11)]) *that*

$$(4.2) \qquad \sum_{k=0}^{n} p_{nk}(u) = 1 \qquad \textit{for } 0 \leqq u \leqq 1 \quad \textit{and } n = 1,2,3,\cdots.$$

We use later the following two results about generalized Bernstein polynomials which are interesting by themselves too.

THEOREM 4.1. *Let $n(\lambda)$, $m(\lambda)$ $(\lambda > 0)$ be positive integral valued functions satisfying $m(\lambda) \leqq n(\lambda)$, $m(\lambda) \to \infty$, $n(\lambda) \to \infty$ $(\lambda \to \infty)$, $\alpha_{n(\lambda),m(\lambda)} \equiv q(\lambda) \to q$ where $0 < q \leqq 1$. Suppose $G(t)$ is continuous and of bounded variation in $[0,1]$ and absolutely continuous in each interval $[\delta, 1-\delta]$ $(0 < \delta < \frac{1}{2})$. Then we have (for $n \equiv n(\lambda)$, $m \equiv m(\lambda)$)*

$$(4.3) \qquad \lim_{\lambda\to\infty} \sum_{k=0}^{n} \left| \int_0^1 \{p_{mk}(u) - p_{nk}(qu)\} dG(u) \right| = 0.$$

In the proof of Theorem (4.1) we use the following results.

LEMMA 4.1. *For a real $b \geqq 1$ we have*

$$(4.4) \qquad \int_0^1 P_{nk}(u) u^{b-1} du = \begin{cases} 0 & \text{if } k > n \\ \dfrac{1}{\lambda_k + b} \left\{ \left(\dfrac{\lambda_n}{\lambda_n + b}\right)! \Big/ \left(\dfrac{\lambda_k}{\lambda_k + b}\right)! \right\} & \text{if } 0 \leqq j \leqq n. \end{cases}$$

and for $0 < k \leqq n$

$$(4.5) \qquad \int_0^1 p_{nk}(u) \frac{du}{u} = \frac{1}{\lambda_k}.$$

Proof. By (2.3) we see that (4.4) is true for $k > n$. Suppose $0 \leqq k \leqq n$. By (2.3) we have

$$\int_0^1 p_{nk}(u)u^{b-1}dx = (-1)^{n-k}\frac{\lambda_n!}{\lambda_k!}\left[\int_0^1 t^{\lambda_k+b-1}dt,\cdots,\int_0^1 t^{\lambda_n+b-1}dt\right]$$
$$= (-1)^{n-k}\frac{\lambda_n!}{\lambda_k!}\left[\frac{1}{\lambda_k+b},\cdots,\frac{1}{\lambda_n+b}\right] .$$

From (2.4) we get immediately by induction

$$(-1)^{n-k}\left[\frac{1}{\lambda_k+b},\cdots\frac{1}{\lambda_n+b}\right] = \frac{1}{\lambda_k+b}\frac{(\lambda_k+b)!}{(\lambda_n+b)!}.$$

Hence (4.4) is true for $0 \leqq k \leqq n$. The proof of (4.5) is similar. Q.E.D.

LEMMA 4.2. *For* $0 \leqq k \leqq m \leqq n$, $m > 0$, *we have*

$$\int_0^1 p_{mk}(u)du \leqq \alpha_{nm}^{-1}\int_0^1 p_{nk}(u)du. \tag{4.6}$$

Proof. For $m = n$ (4.6) is obviously true. Suppose $0 < m < n$. By Lemma (4.1) we have

$$\int_0^1 p_{mk}(u)du - \alpha_{nm}^{-1}\int_0^1 p_{nk}(u)du = \tag{4.7}$$
$$= \frac{1}{\lambda_k+1}\left\{\left(\frac{\lambda_m}{\lambda_m+1}\right)!\Big/\left(\frac{\lambda_k}{\lambda_k+1}\right)!\right\}\left\{1-\frac{\lambda_m!}{y_n!}\right\}$$

where $y_p = \{(1+1/\lambda_p)^{\lambda_1}(1-\lambda_1/\lambda_p)\}^{1/\lambda_1}$. Denote $x_p = 1/\lambda_p$. Choose $M \geqq 1$ such that $\lambda_m > 1$. For $p > m \geqq M$ we have $x_p \leqq 1$, $\lambda_1 x_p \leqq 1$. By computing the derivative of the function $y(x) \equiv (1+x)^{\lambda_1}(1-\lambda_1 x)$ in the interval $[0,\min(1,1/\lambda_1)]$ we see that $y(x)$ is non-increasing in this interval. Thus $y(x) \leqq y(0) = 1$ for $0 \leqq x \leqq \min(1,1/\lambda_1/$. Hence for $x = x_p$ we get $y_p \leqq 1$. Applying the last inequality to the right hand side of (4.7) we get (4.6). Q.E.D.

LEMMA 4.3. *Let $G(u)$ be continuous and of bounded variation in $[0,1]$ and absolutely continuous in each interval $[\delta, 1-\delta]$, $0 < \delta < \frac{1}{2}$. Suppose the sequence $\{q_n\}$ $(n \geqq 0)$ satisfies $0 < q_n \leqq 1$, $q_n \to q$, where $0 < q \leqq 1$. Then we have*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left|\int_0^1 \{p_{nk}(qu) - p_{nk}(q_n u)\}dG(u)\right| = 0. \tag{4.8}$$

Proof. By (4.2) we have for each $0 < \delta < \frac{1}{2}$;

$$I \equiv \sum_{k=0}^{n}\left|\int_0^\delta \{p_{nk}(qu) - p_{nk}(q_n u)\}dG(u)\right| + \sum_{k=0}^{n}\left|\int_{1-\delta}^1 \{p_{nk}(qu) - p_{nk}(q_n u)\}dG(u)\right|$$
$$\leqq 2\int_0^\delta |dB(u)| + 2\int_{1-\delta}^1 |dG(u)|.$$

Hence, since $G(u)$ is continuous at $u = 0$ and $u = 1$,

$$0 \leqq I < \varepsilon \quad \text{if} \quad 0 < \delta < \delta(\varepsilon). \tag{4.9}$$

Choose δ such that $0 < \delta < \delta(\varepsilon)$ The function $G(u)$ is absolutely continuous in $[\delta, 1 - \delta]$, that is, it is an integral of some function $g(u)$, and we have

$$\sum_{k=0}^{n} \left| \int_{\delta}^{1-\delta} \{p_{nk}(qu) - p_{nk}(q_n u)\} dG(u) \right| = \sum_{k=0}^{n} \left| \int_{\delta}^{1-\delta} \{p_{nk}(qu) - p_{nk}(q_n u)\} g(u) du \right|. \tag{4.10}$$

The continuous functions are dense in the space $L[\delta, 1 - \delta]$, hence there exists a continuous function $h(u)$ in $[\delta, 1 - \delta]$ such that

$$\int_{\delta}^{1-\delta} |h(u) - g(u)| du < \varepsilon/2. \tag{4.11}$$

We have

$$\begin{aligned} &\left| \sum_{k=0}^{n} \left| \int_{\delta}^{1-\delta} \{p_{nk}(qu) - p_{nk}(q_n u)\} g(u) du \right| - \right. \\ &\left. - \sum_{k=0}^{n} \left| \int_{\delta}^{1-\delta} \{p_{nk}(qu) - p_{nk}(q_n u)\} h(u) du \right| \right| \leqq \\ &\leqq \sum_{k=0}^{n} \int_{\delta}^{1-\delta} |p_{nk}(qu) - p_{nk}(q_n u)| \, |g(u) - h(u)| du \end{aligned} \tag{4.12}$$

(by (4.2))

$$\leqq 2 \int_{\delta}^{1-\delta} |g(u) - h(u)| du$$

(by (4.11))

$$< \varepsilon.$$

Also for $q_n \leqq q$ (if $q_n \geqq q$ the proof is similar)

$$\begin{aligned} &\sum_{k=0}^{n} \left| \int_{\delta}^{1-\delta} \{p_{nk}(qu) - p_{nk}(q_n u)\} h(u) du \right| = \\ &= \sum_{k=0}^{n} \left| \frac{1}{q} \int_{q\delta}^{q(1-\delta)} p_{nk}(u) h\left(\frac{u}{q}\right) du - \frac{1}{q_n} \int_{q_n\delta}^{q_n(1-\delta)} p_{nk}(u) h\left(\frac{u}{q_n}\right) du \right| \\ &= \sum_{k=0}^{n} \left| \frac{1}{q} \int_{q\delta}^{q_n(1-\delta)} p_{nk}(u) \left\{ h\left(\frac{u}{q}\right) - h\left(\frac{u}{q_n}\right) \right\} du \right. \\ &\quad + \left(\frac{1}{q} - \frac{1}{q_n}\right) \int_{q\delta}^{q_n(1-\delta)} p_{nk}(u) h\left(\frac{u}{q_n}\right) du - \\ &\quad \left. - \frac{1}{q_n} \int_{q_n\delta}^{q\delta} p_{nk}(u) h\left(\frac{u}{q_n}\right) du + \frac{1}{q} \int_{q_n(1-\delta)}^{q(1-\delta)} p_{nk}(u) h\left(\frac{u}{q}\right) du \right| \end{aligned} \tag{4.13}$$

(by (4.2))

$$\leqq \int_{\delta}^{[(1-\delta)q_n]/q} \left| h(u) - h\left(\frac{uq}{q_n}\right) \right| du + \left| \frac{1}{q} - \frac{1}{q_n} \right| \int_{\delta q/q_n}^{(1-\delta)} |h(u)| du +$$

$$+ \frac{1}{q_n} \int_{\delta}^{q\delta/q_n} |h(u)| du + \frac{1}{q} \int_{q_n(1-\delta)/q}^{1-\delta} |h(u)| du$$

(and since $h(u)$ is uniformly continuous in $[\delta, 1-\delta]$ and L-integrals are absolutely continuous)

$$\to 0 \qquad (n \to \infty).$$

By (4.9), (4.10), (4.12) and 4.13) we get (4.8). Q.E.D

Proof of Theorem 4.1. In the first part of the proof we suppose that $G(t)$ is an integral, that is $G(t) = \int_0^t g(u)du$ $0 \leqq t \leqq 1$. Suppose, first, that $g(u)$ is continuous in $[0,1]$. We have

$$\begin{aligned} \sum_{k=0}^{n} \left| \int_0^1 \{p_{mk}(u) - p_{nk}(qu)\} g(u) du \right| &\leqq \sum_{k=0}^{m} \left| \int_0^1 \{p_{mk}(u) - p_{nk}(\alpha_{nm}u)\} g(u) du \right| \\ &+ \sum_{k=m+1}^{n} \left| \int_0^1 p_{nk}(\alpha_{nm}u) g(u) du \right| + \\ &+ \sum_{k=0}^{n} \left| \int_0^1 \{p_{nk}(\alpha_{nm}u) - p_{nk}(qu)\} g(u) du \right| \\ &\equiv I_\lambda^{(1)} + I_\lambda^{(2)} + I_\lambda^{(3)}. \end{aligned} \tag{4.14}$$

By Lemma (4.3) we have

$$\lim_{\lambda \to \infty} I_\lambda^{(3)} = 0. \tag{4.15}$$

By Lemma (3.2) we get, since $g(u)$ is bounded,

$$\lim_{\lambda \to \infty} I_\lambda^{(2)} = 0. \tag{4.16}$$

We have, by (2.6),

$$\begin{aligned} 0 \leqq I_\lambda^{(1)} &\leqq \sum_{k=0}^{m} \left| \int_0^1 p_{mk}(u) g(u) du - g(\alpha_{mk}) \int_0^1 p_{mk}(u) du \right| + \\ &+ \sum_{k=0}^{m} |g(\alpha_{mk})| \left| \int_0^1 P_{mk}(u) du - \frac{1}{\alpha_{nm}} \int_0^1 p_{nk}(u) du \right| \\ &+ \frac{1}{\alpha_{nm}} \sum_{k=0}^{m} \left| \int_0^{\alpha_{nm}} p_{nk}(u) g\left(\frac{u}{\alpha_{nm}}\right) du - g\left(\frac{\alpha_{nk}}{\alpha_{nm}}\right) \int_0^1 p_{nk}(u) du \right| \\ &\equiv I_\lambda^{(11)} + I_\lambda^{(12)} + I_\lambda^{(13)}. \end{aligned} \tag{4.17}$$

By Theorem (3.2) and (2.6) we have

$$\lim_{\lambda\to\infty} I_\lambda^{(11)} = 0. \tag{4.18}$$

Define $g_\lambda^*(u) = g(u/\alpha_{nm})$ $(0 \leqq u \leqq \alpha_{nm})$, $g_\lambda^*(u) = 0$ $(\alpha_{nm} < u \leqq 1)$ and $g^*(u) = g(u/q)$ $(0 \leqq u \leqq q)$, $g^*(u) = 0$ $(q < u \leqq 1)$. We have by (2.6)

$$\begin{aligned} I_\lambda^{(13)} &= \alpha_{nm}^{-1} \sum_{k=0}^{m} \left| \int_0^1 p_{nk}(u) g_\lambda^*(u)du - g_\lambda^*(\alpha_{nk}) \int_0^1 p_{nk}(u)du \right| \\ &\leqq \alpha_{nm}^{-1} \sum_{k=0}^{m} \left| \int_0^1 p_{nk}(u) g^*(u)du - g^*(\alpha_{nk}) \int_0^1 p_{nk}(u)du \right| \\ &\quad + \alpha_{nm}^{-1} \int_0^1 \left\{ \sum_{k=0}^{n} p_{nk}(u) \right\} \left| g_\lambda^*(u) - g^*(u) \right| du \\ &\quad + \alpha_{nm}^{-1} \sum_{k=0}^{m} \left| g^*(\alpha_{nk}) - g_\lambda^*(\alpha_{nk}) \right| \int_0^1 p_{nk}(u)du \\ &\equiv I_\lambda^{(131)} + I_\lambda^{(132)} - I_\lambda^{(133)}. \end{aligned} \tag{4.19}$$

By Theorem (3.2) we have

$$\lim_{\lambda\to\infty} I_\lambda^{(131)} = 0. \tag{4.20}$$

By (4.2) and Lebesgue's dominated convergence theorem we have

$$\lim_{\lambda\to\infty} I_\lambda^{(132)} = \frac{1}{q} \lim_{\lambda\to\infty} \int_0^1 \left| g_\lambda^*(u) - g^*(u) \right| du = \frac{1}{q} \int_0^1 0\, du = 0. \tag{4.21}$$

Given $\varepsilon > 0$ we have $q - \varepsilon \leqq \alpha_{nm} \leqq q + \varepsilon$ (or $q - \varepsilon \leqq \alpha_{nm} \leqq q$, if $q = 1$) for $\lambda > \Lambda_4(\varepsilon)$. Hence by (4.2)

$$0 \leqq I_\lambda^{(133)} \leqq \frac{2K}{\alpha_{nm}} \int_0^1 \left\{ \sum_{\substack{k \\ q-\varepsilon \leqq \alpha_{nk} \leqq q+\varepsilon}} p_{nk}(u) \right\} du + \max_{\substack{k \\ \alpha_{nk} \leqq q-\varepsilon}} \left| g(\alpha_{nk}/q) - g(\alpha_{nk}/\alpha_{nm}) \right|.$$

By Theorem (4.I), Lebesgue's dominated convergence theorem and the uniform continuity of $g(u)$ in $[0,1]$ we get

$$0 \leqq I_\lambda^{(133)} \leqq 2K \int_{q-\varepsilon}^{q+\varepsilon} du + o(1) \qquad (\lambda \to \infty).$$

Hence

$$\lim_{\lambda\to\infty} I_\lambda^{(133)} = 0. \tag{4.22}$$

By (4.19), (4.20), (4.21) and (4.22) we get

$$\lim_{\lambda\to\infty} I_\lambda^{(13)} = 0. \tag{4.23}$$

By Lemma (4.2) and (4.2) we have for $\lambda > \Lambda_4(\varepsilon)$, $\varepsilon > 0$,

$$0 \leqq I_\lambda^{(12)} = \sum_{k \infty 0}^{m} | g(\alpha_{mk}) | \left\{ \frac{1}{\alpha_{nm}} \int_0^1 p_{nk}(u)du - \int_0^1 p_{mk}(u)du \right\}$$

$$\leqq K \sum_{k=0}^{m} \left\{ \frac{1}{\alpha_{nm}} \int_0^1 p_{nk}(u)du - \int_0^1 p_{mk}(u)du \right\}$$

$$= K \left\{ \frac{1}{\alpha_{nm}} \int_0^1 \sum_{\substack{k \\ \alpha_{nk} \leqq \alpha_{nm}}} p_{nk}(u)du - 1 \right\}$$

$$\leqq K \left\{ \frac{1}{\alpha_{nm}} \int_0^1 \sum_{\substack{k \\ \alpha_{nk} \leqq q+\varepsilon}} p_{nk}(u)du - 1 \right\}$$

(by Lemma (3.2))

$$\rightarrow K \left\{ \frac{1}{q} \int_0^{q+\varepsilon} du - 1 \right\} \qquad (\lambda \rightarrow \infty)$$

$$= \frac{K\varepsilon}{q}.$$

Letting $\varepsilon \downarrow 0$ we get

$$\lim_{\lambda \to \infty} I_\lambda^{(12)} = 0. \tag{4.24}$$

By (4.17), (4.18), (4.23) and (4.24) we get

$$\lim_{\lambda \to \infty} I_\lambda^{(1)} = 0. \tag{4.25}$$

By (4.14), (4.15), (4.16) and (4.25) we see that our theorem is true for $g(u) \in C[0,1]$. The continuous functions on $[0,1]$ are dense in the space $L[0,1]$, hence, as in the proof of Lemma (4.3) we see that our theorem is true for $g(u) \in L[0,1]$. Suppose now that $G(t)$ is continuous and of bounded variation in $[0,1]$ and absolutely continuous in each interval $[\delta, 1-\delta]$ $(0 < \delta < \frac{1}{2})$. The rest of the proof follows now by estimating the integrals in (4.3) over $[0,\delta]$ and $[1-\delta, 1)$ as in the proof of Lemma (4.3) and applying the first part of the proof to the the integrals over $[\delta, 1-\delta]$. Q.E.D.

5. Proof of Theorems (2.1) and (2.2).

In the proof of Theorems (2.1) and (2.2) we use the following results about generalized Hausdorff transformations.

LEMMA 5.1. *For a fixed sequence* $\{\lambda_n\}$ $(n \geqq 0)$ *of different numbers we have for the divided differences of any two sequences* $\{\mu_n^{(1)}\}$, $\{\mu_n^{(2)}\}$ *in respect to* $\{\lambda_n\}$

$$[\mu_p^{(3)},\cdots,\mu_n^{(3)}] = \sum_{k=p}^{n} [\mu_p^{(1)},\cdots,\mu_k^{(1)}][\mu_k^{(2)},\cdots,\mu_n^{(2)}] \quad \text{for} \quad 0 \leqq p \leqq n,$$

where $\mu_n^{(3)} = \mu_n^{(1)}\mu_n^{(2)}$.

Proof. The proof follows by the definition of the divided differences and by induction on n, $n \geqq p \geqq 0$ since the result is true for $n = p \geqq 0$. Q.E.D.

LEMMA 5.2. *Suppose* $\beta(t)$ *is a normalized function of bounded variation in* $[0,1]$ *satisfying* $\beta(1) = 1$. *Denote* $\mu_k = \int_0^1 t^{\lambda_k} d\beta(t)$ $(k \geqq 0)$. *We have* (i) *for* $0 \leqq \alpha \leqq 1$ *and* $0 \leqq p \leqq k$

$$\int_0^1 p_{kp}(u) d\Psi_\alpha(u) = (-1)^{k-p} \frac{\lambda_k!}{\lambda_p!} \left[\frac{1}{\binom{\lambda_p+\alpha}{\lambda_-}},\cdots,\frac{1}{\binom{\lambda_n+\alpha}{\lambda_n}}\right],$$

where $\Psi_0(u) = 0$, $(0 \leqq u < 1)$, $\Psi_0(1) = 1$, $\Psi_\alpha(u) = 1-(1-u)^\alpha$ $(\alpha > 0)$, $0 \leqq u \leqq 1$. (ii) We have for $0 < k \leqq n$

$$(-1)^{n-k} \frac{\lambda_n!}{\lambda_k!} \left[\binom{\lambda_k+\alpha}{\lambda_k}\frac{\mu_k}{\lambda_k},\cdots,\binom{\lambda_n+\alpha}{\lambda_n}\frac{\mu_n}{\lambda_n}\right]$$

$$= \begin{cases} \dfrac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \displaystyle\int_0^1 \frac{1-\beta(t)}{t}\left\{\int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dx\right\}dt & \\ & \text{if } 0 \leqq \alpha < 1 \\ \displaystyle\int_0^1 \frac{1-\beta(t)}{t}[p_{nk}(t)t]'dt & \text{if } \alpha = 1. \end{cases}$$

Proof. Suppose $0 \leqq \alpha < 1$. For $0 \leqq t \leqq 1$ and $0 < k \leqq n$ we have by (2.3)

$$\int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dx$$

$$= (-1)^{n-k}\frac{\lambda_n!}{\lambda_k!}\int_0^t (t-x)^{-\alpha}[(\lambda_k+\alpha)x^{\lambda_k+\alpha-1},\cdots,(\lambda_n+\alpha)x^{\lambda_n+\alpha-1}]dx$$

$$= (-1)^{n-k}\frac{\lambda_n!}{\lambda_k!}\left[(\lambda_k+\alpha)\int_0^t (t-x)^{-\alpha}x^{\lambda_k+\alpha-1}dx,\cdots,(\lambda_n+\alpha)\int_0^t (t-x)^{-\alpha}x^{\lambda_n+\alpha-1}dx\right]$$

$$= \Gamma(\alpha+1)\Gamma(1-\alpha)\cdot(-1)^{n-k}\frac{\lambda_n!}{\lambda_k!}\left[\binom{\lambda_k+\alpha}{\lambda_k}t^{\lambda_k},\cdots,\binom{\lambda_n+\alpha}{\lambda_-}t^{\lambda}\right].$$

Multiplying both sides by $[1-\beta(t)]/t$ and integrating over $[0,1]$ we get the second conclusion for $0 \leqq \alpha < 1$. For $\alpha = 1$ the proof is the same. The proof of the first consequence is similar. Q.E.D.

LEMMA 5.3. *For* $0 < k \leqq n$ *we have* $\left[\frac{1}{\lambda_k}, \cdots, \frac{1}{\lambda_n}\right] = (-1)^{n-k}(\lambda_k \cdots \lambda_n)^{-1}$.

Proof. The proof follows from (2.4) by induction.

LEMMA 5.4. *Suppose* $0 \leqq \alpha < 1$. *Let* $\beta(t)$ *be a normalized function of bounded variation in* $[0,1]$ *satisfying* $\beta(1) = 1$. *Then for* $n \geqq 0$ *we have*

$$(H_n(\beta)) = \begin{cases} a_0 + \dfrac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \displaystyle\sum_{k=1}^{n} a_k^{(\alpha)} \int_0^1 \frac{1-\beta(t)}{t} \left\{ \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]' dx \right\} dt & 0 \leqq \alpha < 1 \\ a_0 + \displaystyle\sum_{k=1}^{n} a_k^{(1)} \int_0^1 \frac{1-\beta(t)}{t} [p_{nk}(t)t]' dt & \alpha = 1. \end{cases}$$

Proof. Suppose $0 \leqq \alpha < 1$. We have for $n > 0$

$$a_0 + \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \sum_{k=1}^{n} a_k^{(\alpha)} \int_0^1 \frac{1-\beta(t)}{t} \left\{ \int_0^t (1-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]' dx \right\} dt$$

$$= a_0 + \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \sum_{k=1}^{n} \left[\int_0^1 \sum_{p=1}^{k} p_{kp}(u)\lambda_p a_p d\Psi_\alpha(u) \right] \int_0^1 \frac{1-\beta(t)}{t} \times$$

$$\times \left\{ \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]' dx \right\} dt$$

(changing the order of summation)

$$= a_0 + \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)} \sum_{p=1}^{n} \lambda_p a_p \sum_{k=p}^{n} \left[\int_0^1 p_{kp}(u) d\Psi_\alpha(u) \right] \int_0^1 \frac{1-\beta(t)}{t} M \times$$

$$\times \left\{ \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]' dx \right\} dt$$

(by Lemma (5.2))

$$= a_0 + \sum_{p=1}^{n} \lambda_p a_p (-1)^{n-p} \frac{\lambda_n!}{\lambda_k!} \sum_{k=p}^{n} \left[\binom{\lambda_n + \alpha}{\lambda p}^{-1}, \cdots, \binom{\lambda_k + \alpha}{\lambda_k}^{-1} \right] \left[\binom{\lambda_k + \alpha}{\lambda k} \frac{\mu_k}{\lambda_k}, \cdots, \binom{\lambda_n + \alpha}{\lambda} \frac{\mu_n}{\lambda_n} \right]$$

(by Lemma (5.1))

$$= a_0 + \sum_{p=1}^{n} \lambda_p a_p (-1)^{n-p} \frac{\lambda_n!}{\lambda_p!} \left[\frac{\mu_p}{\lambda_p}, \cdots, \frac{\mu_n}{\lambda_n} \right]$$

(by Lemma (5.1), (5.3) and by changing the order of summation)

$$= a_0 + \sum_{k=1} (-1)^{n-k} \frac{\lambda_n!}{\lambda_k!} [\mu_k, \cdots, \mu_n] \sum_{p=1}^{n} a$$

$$= H_n(\beta).$$

This proves our theorem for $0 \leqq \alpha < 1$. The proof for $\alpha = 1$ is similar.

LEMMA 5.5. *Let $\beta(t)$ be a normalized function of bounded variation in $[0,1]$ satisfying $\beta(1) = 1$. Then for $0 < k \leqq n$ we have*

$$\int_0^1 [p_{nk}(t)t]' \frac{1-\beta(t)}{t} dt = - \int_0^1 p_{nk}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] + \frac{1}{\lambda_k},$$

and

$$\int_0^1 [p_{nk}(t)t]' \frac{1-\beta(t)}{t} dt = \int_0^1 p_{nk}(t) d\left[\int_t^1 u d\left(\frac{1-\beta(u)}{u}\right)\right].$$

Proof. We have

$$\int_0^1 [p_{nk}(t)t]' \frac{1-\beta(t)}{t} dt = \int_0^1 p'_{nk}(t)[1-\beta(t)]dt + \int_0^1 p_{nk}(t) \frac{1-\beta(t)}{t} dt$$

(integrating by parts)

$$= - \int_0^1 p_{nk}(t) d[1-\beta(t)] - \int_0^1 p_{nk}(t) \frac{\beta(t)}{t} dt + \int_0^1 p_{nk}(t) \frac{dt}{t}$$

(by (4.5))

$$= - \int_0^1 p_{nk}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] + \frac{1}{\lambda_k}.$$

The proof of the second result is the same. Q.E.D.

LEMMA 5.6. *If $n(\lambda)$, $m(\lambda)$ are positive integral valued functions satisfying (2.5), $m(\lambda) < n(\lambda)$, and $\{\lambda_n\}$ satisfies (3.1), and $\alpha_{n(\lambda),m(\lambda)} \to q$ then*

$$\lim_{\lambda \to \infty} \sum_{k=m+1}^{n} \frac{1}{\lambda_k} = \log \frac{1}{q}$$

Lemma (5.6) is Lemma 2 of [10].

The following results are quite easily proved.

LEMMA 5.7. *Suppose $\gamma(u)$ is a function of bounded variation in $[0,1]$. Then for each ρ, $0 < \rho < 1$, the function $F_\rho(u)$ defined by*

$$F_\rho(0) = \gamma(0+) \int_1^\infty \frac{dy}{y(y-1)^\rho}, \qquad F_\rho(1) = 0.$$

$$F_\rho(u) = u^\rho \int_u^1 \frac{\gamma(v)}{v} (v-u)^{-\rho} dv \quad (0 < u < 1)$$

is continuous and of bounded variation in $[0,1]$, *absolutely continuous in each interval* $[\delta,1]$ $(0<\delta<1)$ *and*

$$\frac{1}{1-\rho}\int_t^1 (v-t)^{1-\rho} d\left(\frac{\gamma(v)}{v}\right) = \int_t^1 \left[\int_u^1 (v-u)^{-\rho} d\left(\frac{\gamma(v)}{v}\right)\right] du.$$

LEMMA 5.8. *Suppose* $\gamma(t)$ *is a function of bounded variation in* $[0,1]$ *and* $\gamma(t)/t \in L[0,1]$. *Then for each* α, $0 \leqq \alpha < 1$, *the function*

$$F(x) = \int_x^1 t^\alpha d_t\left(\int_t^1 (u-t)^{-\alpha}\frac{\gamma(u)}{u}du\right)$$

is absolutely continuous in $[0,1]$.

LEMMA (5.9). *For fixed* α, q, $0 \leqq \alpha < 1$, $0 < q \leqq 1$, *the function* $F(x)$ *defined by*

$$F(x) = \int_q^1 \frac{(t-x)^{-\alpha}}{t}dt \ (0 \leqq x \leqq q), \quad F(x) = F(q) \qquad (q < x \leqq 1)$$

is continuous and of bounded variation in $[0,1]$ *and absolutely continuous in each interval* $[\delta,1]$ $(0<\delta<1)$. *Also, the function* $G(x) \equiv \int_0^x t^\alpha dF(t)$ *is continuous and of bounded variation in* $[0,1]$ *and absolutely continuous in each interval* $[\delta,1]$ $(0<\delta<1)$.

LEMMA 5.10. *For each* $\alpha, 0 \leqq \alpha < 1$, *the function* $H(x) \equiv \frac{1}{x}\int_0^x (1-u)^{-\alpha}u^{\alpha-1}du$ *is Lebesgue integrable in* $[0,1]$ *and continuous in each interval* $[\delta,1]$ $(0<\delta<1)$.

Proof of Theorem 2.1. For $n \geqq m > 0$ we have by Lemma (5.4)

(5.2) $H_n(\beta) - H_m(\gamma) =$

$$= \sum_{k=1}^m a_k^{(\alpha)}(\Gamma(1-\alpha)\Gamma(1+\alpha))^{-1}\left\{\int_0^1 \frac{1-\beta(t)}{t}\int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dxdt\right.$$

$$\left. - \int_0^1 \frac{1-\gamma(t)}{t}\int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dxdt\right\}$$

$$+ \sum_{k=m+1}^n a_k^{(\alpha)}(\Gamma(1-\alpha)\Gamma(1+\alpha))^{-1}\int_0^1 \frac{1-\beta(t)}{t}\int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dxdt.$$

Changing the order of integration and then integrating by parts, using Lemmas (5.7) and (5.8) and the fact that Lebesgue integrals do not depend on the value of the integrand at one point, we get

$$\int_0^1 \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^\alpha]'dxdt$$

$$(5.3) \qquad = -\int_0^1 p_{nk}(x)d_x\left\{\int_x^1 t^\alpha d_t\left(\int_t^1 (u-t)^{-\alpha}\frac{\beta(u)}{u}du\right)\right\}$$

$$-\int_0^1 p_{nk}(x)x^\alpha d_x\left(\int_x^1 \frac{(t-x)^{-\alpha}}{t}dt\right)$$

(and making the substitution $ut = x$ in the last integral)

$$= -\int_0^1 p_{nk}(x)d_x\left\{\int_x^1 t^\alpha d_t\left(\int_t^1 (u-t)^{-\alpha}\frac{\beta(u)}{u}du\right\} + \int_0^1 p_{nk}(x)(1-x)^{-\alpha}x^{\alpha-1}dx$$

$$+\alpha\int_0^1 p_{nk}(x)x^{-1}\left(\int_x^1 (1-y)^{-\alpha}u^{\alpha-1}du\right)dx.$$

By simple computation we get $0<\alpha<1$

$$\int_0^1 p_{nk}(x)x^\alpha d_x\left(\int_x^1 \frac{(t-x)^{-\alpha}}{t}dt\right) - \int_0^1 p_{mk}(x)x^\alpha d_x\left(\int_x^1 \frac{(t-x)^{-\alpha}}{t}dt\right)$$

$$= \int_q^1 p_{nk}(x)x^\alpha d_x\left(\int_x^1 \frac{(t-x)^{-\alpha}}{t}dt\right) + \int_0^q p_{nk}(x)x^\alpha d_x\left(\int_q^1 \frac{(1-x)^{-\alpha}}{t}dt\right)$$

$$(5.4) \qquad -\int_0^1 [p_{nk}(qx)-p_{mk}(x)](1-x)^{-\alpha}x^{\alpha-1}dx -$$

$$-\int_0^1 [p_{nk}(qx)-p_{mk}(x)]\left(\frac{\alpha}{x}\int_0^1 (1-u)^{-\alpha}u^{\alpha-1}du\right)dx$$

$$+\int_0^1 [p_{nk}(qx)-p_{mk}(x)]\left(\frac{\alpha}{x}\int_0^x (1-u)^{-\alpha}u^{\alpha-1}du\right)dx.$$

By Lemma (4.1) we have for $0<k\leqq m$

$$\int_0^1 [p_{nk}(qx)-p_{mk}(x)]\frac{dx}{x} = \int_0^1 p_{nk}(qx)\frac{dx}{x} - \int_0^1 p_{mk}(x)\frac{dx}{x}$$

$$(5.5) \qquad = \int_0^q p_{nk}(x)\frac{dx}{x} - \int_0^1 p_{nk}(x)\frac{dx}{x}$$

$$= -\int_q^1 p_{nk}(x)\frac{dx}{x}.$$

By (5.3), (5.4) and (5.5) we have for $0 < \alpha < 1$

$$\int_0^1 \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt - $$

$$- \int_0^1 \frac{1-\gamma(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{mk}(x)x^{\alpha}]'dxdt$$

$$(5.6a) \qquad = - \int_0^1 p_{nk}(x)d_x\left\{\int_x^1 t^{\alpha}d_t\left(\int_t^1 (u-t)^{-\alpha}\frac{\beta(u)}{u}du\right)\right\}$$

$$+ \int_0^1 p_{mk}(x)d_x\left\{\int_x^1 t^{\alpha}d_t\left(\int_t^1 (u-t)^{-\alpha}\frac{\gamma(u)}{u}du\right)\right\}$$

$$- \int_0^q p_{nk}(x)d_x\left\{\int_0^x t^{\alpha}d_t\left(\int_q^1 \frac{(u-t)^{-\alpha}}{u}du\right)\right\} - \Gamma(1-\alpha)\Gamma(1+\alpha)\int_q^1 p_{nk}(x)d\left(\log\frac{x}{q}\right)$$

$$- \int_q^1 p_{nk}(x)x^{\alpha}d_x\left(\int_x^1 \frac{(t-x)^{-\alpha}}{t}dt\right) + \int_0^1 [p_{nk}(qx) - p_{mk}(x)](1-x)^{-\alpha}x^{\alpha-1}dx$$

$$- \int_0^1 [p_{nk}(qx) - p_{mk}(x)]\left(\frac{\alpha}{x}\int_0^x (1-u)^{-\alpha}u^{\alpha-1}du\right)dx.$$

For $\alpha = 0$ we get by Lemma (4.1)

$$\int_0^1 \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt -$$

$$(5.6b) \qquad - \int_0^1 \frac{1-\gamma(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{mk}(x)x^{\alpha}]'dxdt$$

$$= \int_0^1 p_{nk}(x)d\left(\int_x^1 \frac{\beta(u)}{u}du\right) - \int_0^1 p_{mk}(x)\,d\left(\int_x^1 \frac{\gamma(u)}{u}du\right).$$

For a fixed k, $0 < k \leqq m$, we have by (5.6a) or (5.6b), by Lemma (3.2) (since for a fixed $k > 0$ and a given δ, $0 < \delta < 1$, we have for $n > N(\delta)$ $\alpha_{nk} < \delta$), by Lemma (5.8), Lemma (5.9), Lemma (4.3) and Lemma (5.10)

$$(5.7) \qquad \lim_{\lambda\to\infty}\left\{\int_0^1 \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt - \right.$$

$$\left. - \int_0^1 \frac{1-\gamma(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{mk}(x)x^{\alpha}]'dxdt \right\} = 0$$

for $k = 1,2,\cdots$.

We have by Theorem (4.1) and Lemma (5.10) for $0 < \alpha < 1$

$$\lim_{\lambda\to\infty} \sum_{k=1}^{m} \left| \int_0^1 [p_{nk}(qx) - p_{mk}(x)](1-x)^{-\alpha}x^{\alpha-1}dx - \right. \tag{5.8}$$

$$\left. - \int_0^1 [p_{nk}(qx) - p_{mk}(x)] \left(\frac{\alpha}{x} \int_0^x (1-u)^{-\alpha}u^{\alpha-1}du\; dx \right| = 0.$$

Now by Theorem (3.1), (5.6a) or (5.6b), (5.8), Lemma (5.8) and Lemma (5.9) we get

$$\lim_{\lambda\to\infty} \sum_{k=1}^{m} \left| \int_0^1 \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt \right.$$

$$\left. - \int_0^1 \frac{1-\gamma(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt \right| \tag{5.9}$$

$$= \int_0^q \left| d_x \left\{ \int_x^1 t^{\alpha} d_t \left(\int_t^1 (u-t)^{-\alpha} \frac{\beta(u)}{u} du \right) + \int_0^x t^{\alpha} d_t \left(\int_q^1 \frac{(u-t)^{-\alpha}}{u} du \right) \right.\right.$$

$$\left.\left. - \int_{x/q}^1 t^{\alpha} d_t \left(\int_t^1 (u-t)^{-\alpha} \frac{\gamma(u)}{u} du \right) \right\} \right|.$$

We have by (5.3) for $n > m > 0$ and $0 < \delta < q$

$$\sum_{k=m+1}^{n} \left| \int_0^{\delta} \frac{1-\beta(t)}{t} \int_0^t (t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt \right|$$

$$\leqq \sum_{k=m+1}^{n} \left| \int_0^{\delta} p_{nk}(x) d_x \left\{ \int_x^1 t^{\alpha} d_t \left(\int_t^1 (u-t)^{-\alpha} \frac{\beta(u)}{u} du \right) \right\} \right| \tag{5.10}$$

$$+ \sum_{k=m+1}^{n} \int_0^{\delta} p_{nk}(x)(1-x)^{-\alpha}x^{\alpha-1}dx + \alpha \sum_{k=m+1}^{n} \int_0^{\delta} p_{nk}(x) \frac{1}{x} \left(\int_x^1 (1-u)^{-\alpha}u^{\alpha-1}du \right) dx$$

$$\equiv J_{\lambda}^{(1)} + J_{\lambda}^{(2)} + J_{\lambda}^{(3)}.$$

By Lemma (5.8) and Lemma (3.2) we get (as in the proof of (5.7))

$$\lim_{\lambda\to\infty} J_{\lambda}^{(1)} = 0. \tag{5.11}$$

By Lemma (3.2) we get

$$\lim_{\lambda\to\infty} J_{\lambda}^{(2)} = 0. \tag{5.12}$$

By [8, (3.12), (3.13), (3.14)] we get

$$\lim_{\lambda\to\infty} J_{\lambda}^{(3)} = 0. \tag{5.13}$$

Thus by (5.10), (5.11), (5.12) and (5.13) we have for each δ, $0<\delta<q$,

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^{\delta}\frac{1-\beta(t)}{t}\int_0^t(t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt\right| = 0. \tag{5.14}$$

Hence by [8, Theorem 2.2, (ii)], (5.14), (5.3), Lemma (5.8) we get

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^{1}\frac{1-\beta(t)}{t}\int_0^t(t-x)^{-\alpha}[p_{nk}(x)x^{\alpha}]'dxdt\right| = \tag{5.15}$$

$$= \int_q^1 x^{\alpha}\left|\, d_x\left(\int_x^1(t-x)^{-\alpha}\,\frac{1-\beta(t)}{t}\,dt\right|\right. .$$

By Agnew's theorem (see [8]), (5.2), (5.7), (5.8) and (5.15) we get the proof of our theorem. Q.E.D.

Proof of Theorem (2.2): By Lemmas (5.4) and (5.5) we have for $0<m\leqq n$

$$H_n(\beta)-H_m(\gamma) =$$

$$= -\sum_{k=1}^{m}a_k^{(1)}\left\{\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]-\int_0^1 p_{mk}(t)d\left[\int_t^1 ud\left(\frac{\gamma(u)}{u}\right)\right]\right\} \tag{5.16}$$

$$+\sum_{k=m+1}^{n}a_k^{(1)}\int_0^1 p_{nk}(t)\,d\left[\int_t^1 ud\left(\frac{1-\beta(u)}{u}\right)\right].$$

Suppose now $0<q<1$ or $q=1$ and $m(\lambda)<n(\lambda)$ for $\lambda>\Lambda$. By [8, (3.26)] and the argument used there we get for $n\equiv n(\lambda)$, $m\equiv m(\lambda)$

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{1-\beta(u)}{u}\right)\right]\right| \tag{5.17'}$$

$$=\sum_q^1\left[\int_t^1 ud\left(\frac{1-\beta(u)}{u}\right)\right]=\int_q^1 u\left|\,d\left(\frac{1-\beta(u)}{u}\right)\right|$$

(where, if $q=1$ and $m(\lambda)<n(\lambda)$ ($\lambda>\Lambda$, then $\int_t^1 u\left|d\left(\frac{1-\beta(u)}{u}\right)\right| = |1-\beta(1-0)|$).

We have

$$\begin{cases}\displaystyle\int_t^1 ud\left(\frac{\beta(u)}{u}\right) = 1-\beta(t)-\int_t^1\frac{\beta(u)}{u}du\\[2ex] \displaystyle\int_t^1 ud\left(\frac{\gamma(u)}{u}\right) = 1-\gamma(y)-\int_t^1\frac{\gamma(u)}{u}du.\end{cases} \tag{5.18}$$

For a fixed $k>0$ we have $\lim_{\lambda\to\infty}\alpha_{nk}=0$. Hence for each $\delta>0$ we have for $n>N(\delta)$ $\alpha_{nk}<\delta$ and by Lemma (3.2) and (5.18) we get

$$\left| \int_0^1 p_{nk}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] \right| \leqq \int_0^1 p_{nk}(t) \left| d\left[1-\beta(t) - \int_t^1 \frac{\beta(u)}{u} du\right]\right|$$

(5.19)
$$\leqq \sum_{\substack{k \\ x_{-k} \leqq \delta}} \int_0^1 p_{nk}(t) \left| d[1-\beta(t) - \int_t^1 \frac{\beta(u)}{u} du \right|$$
$$\to \int_0^\delta \left| d\left[1-\beta(t) - \int_t^1 \frac{\beta(u)}{u} du\right] \right|$$
$$\to 0, \quad \text{as } \delta \downarrow 0.$$

Hence we have for $k = 1, 2, \cdots$

$$\lim_{\lambda\to\infty} \left\{ \int_0^1 p_{nk}(t)\, d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] - \int_0^1 p_{mk}(t) d\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right]\right\} = 0. \tag{5.20}$$

By (2.3) we have $p_{nm}(t) = t^{\lambda_m}$ and $p_{mk}(1) = 0$ for $0 \leqq k < m$.

Also, the function $\gamma_1(t)$ is continuous at $t = 1$ and $\gamma(t) = \gamma_1(t) + \gamma_2(t)$ where $\gamma_2(t) = 0$ $(0 \leqq t < 1)$, $\gamma_2(1) = \gamma(1) - \gamma(1-0)$. Hence

$$\sum_{k=1}^{m} \left| \int_0^1 p_{nk}(t)\, d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right) - \int_0^1 p_{mk}(t) d\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] \right|$$

(5.21)
$$= \sum_{k=0}^{m} \left| \int_0^1 p_{nk}(t)\, d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] - \int_0^1 p_{mk}(t) d\left[\int_t^1 u d\left(\frac{\gamma_1(u)}{u}\right)\right] \right|$$
$$- \left| \int_0^1 p_{nm}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] - \int_0^1 p_{mm}(t) d\left[\int_t^1 u d\left(\frac{\gamma_1(u)}{u}\right)\right] \right|$$
$$+ \left| \int_0^1 p_{nm}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] - \int_0^1 p_{mm}(t) d\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] \right|$$
$$- \left| \int_0^1 p_{n0}(t) d\left[\int_t^1 u d\left(\frac{\beta(u)}{u}\right)\right] - \int_0^1 p_{m0}(t) d\left[\int_t^1 u d\left(\frac{\gamma(u)}{u}\right)\right] \right| .$$

If the function $\gamma(t)$ is absolutely continuous in each interval $[\delta, 1-\delta](0 < < \delta < \frac{1}{2})$ and $\int_0^1 \frac{|\gamma(t)|}{t} dt < \infty$, then by Theorem (4.1) we have

$$\lim_{\lambda\to\infty} \sum_{k=0}^{n} \left| \int_0^1 \{p_{mk}(t) - p_{nk}(qt)\} d\left[\int_t^1 u d\left(\frac{\gamma_1(u)}{u}\right)\right] \right| = 0. \tag{5.22}$$

If, in addition $\int_0^1 \frac{|\beta(t)|}{t} dt < \infty$ and the function $\beta(t)$ is continuous at $t = q$, then by the first alternative in the second conclusion of Theorem (3.1) we get

$$\lim_{\lambda\to\infty}\sum_{k=0}^{m}\left|\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]-\int_0^1 p_{mk}(t)d\left[\int_t^1 ud\,\frac{\gamma_1(u)}{u}\right)\right]\right|$$

$$(5.23)\qquad =\int_0^q\left|d\left\{\int_{t/q}^1 ud\left(\frac{\gamma_1(u)}{u}\right)-\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right\}\right|$$

$$=\int_0^q u\left|d\left(\frac{\beta(u)-\gamma_1\left(\frac{u}{q}\right)}{u}\right)\right|.$$

The functions $\beta(t)$ and $\gamma_1(t)$ are continuous at $t=0$, since the $H_n(\beta)$ and $H_m(\beta)$ transforms are regular. Hence by Lemma (3.2) we have

$$(5.24)\qquad \lim_{\lambda\to\infty}\left|\int_0^1 p_{n0}(t)\,d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]-\int_0^1 p_{m0}(t)d\left[\int_t^1 ud\left(\frac{\gamma_1(u)}{u}\right)\right]\right|=0.$$

Choose $\varepsilon_1>0$, $\varepsilon_2>0$ such that $\beta(t)$ is continuous at the points $q-\varepsilon_1$, $q+\varepsilon_2\leqq 1$. We have by Lemma (3.2), since $\beta(t)$ is continuous at $t=q$

$$\left|\int_0^1 p_{nm}(t)d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]\right|$$

$$(5.25)\qquad \leqq\left\{\sum_{\substack{k\\ \alpha_{nk}\leqq q+\varepsilon_2}}-\sum_{\substack{k\\ \alpha_{nk}\leqq q-\varepsilon_1}}\right\}\int_0^1 p_{nm}(t)\left|d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right.\right|$$

$$\to\int_{q-\varepsilon_1}^{q+\varepsilon_2}\left|d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]\right|$$

$$\to 0\quad\text{as}\quad \varepsilon_1\downarrow 0,\ \ \varepsilon_2\downarrow 0.$$

Hence

$$(5.26)\qquad \lim_{\lambda\to\infty}\int_0^1 p_{nm}(t)d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]=0.$$

Now by Lebesgue's dominated convergence theorem for Lebesgue-Stieltjes integrals we have, since $\gamma_1(t)$ is continuous at $t=1$

$$(5.27)\qquad \lim_{\lambda\to\infty}\int_0^1 p_{mm}(t)d\left[\int_t^1 ud\left(\frac{\gamma_1(u)}{u}\right)\right]=\int_0^1(\lim_{\lambda\to\infty} t^{\lambda_m})d\left[\int_t^1 ud\left(\frac{\gamma_1(u)}{u}\right)\right]$$
$$=0.$$

Also

$$(5.28)\qquad \lim_{\lambda\to\infty}\int_0^1 p_{mm}(t)d\left[\int_t^1 ud\left(\frac{\gamma(u)}{u}\right)\right]=\int_0^1(\lim_{\lambda\to\infty} t^{\lambda_m})d\left[\int_t^1 ud\left(\frac{\gamma(u)}{u}\right)\right]$$
$$=\gamma(1-0)-1.$$

Now by (5.21), (5.23), (5.24), (5.26), (5.27) and (5.28) we get

$$\lim_{\lambda\to\infty}\sum_{k=1}^{m}\left|\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{\beta(u)}{u}\right)\right]-\int_0^1 p_{mk}(t)d\left[\int_t^1 ud\left(\frac{\gamma(u)}{u}\right)\right]\right| =$$

(5.29)

$$=\int_0^1 u\left|d\left(\frac{\beta(u)-\gamma_1\left(\frac{u}{q}\right)}{u}\right)\right|+\left|1-\gamma(1-0)\right|.$$

By Agnew's theorem (see [8]) ,(5.16), (5.20), (5.17) and (5.23) we get the proof of our theorem for assumption (a) and $0 < q < 1$. For $q = 1$ and $m(\lambda) < n(\lambda)$ $(\lambda > \Lambda)$ the only difference in the proof is that we have to show that for $m \equiv m(\lambda)$, $n \equiv n(\lambda)$ we have now

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\int_0^1 ud\left(\frac{1-\beta(u)}{u}\right)\right]\right| = \left|1-\beta(1-0)(1-0)\right|. \tag{5.30}$$

We have for $\beta(t) \equiv \beta_1(t)+\beta_2(t)$

$$\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{1-\beta(u)}{u}\right)\right]\right| \tag{5.31}$$

$$=\sum_{k=m+1}^{n}\left|-\int_0^1 p_{nk}(t)d\left[\beta_1(1)-\beta_1(t)+\int_0^t\frac{\beta_1(u)}{u}du\right]+\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{1-\beta_2(u)}{u}\right)\right]\right|$$

$$=\left\{\sum_{k=0}^{n}-\sum_{k=0}^{m}\right\}\left|-\int_0^1 p_{nk}(t)d\left[\beta_1(1)-\beta_1(t)+\int_0^t\frac{\beta_1(u)}{u}du\right]+\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{1-\beta_2(u)}{u}\right)\right]\right|$$

By the second conclusion of Theorem (3.1) for $s_0(t) = s_m(t) \equiv 0$ we get

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\beta_1(1)-\beta_1(t)+\int_0^t\frac{\beta_1(u)}{u}du\right]\right| = 0. \tag{5.32}$$

By (5.31) and (5.32) we get now

$$\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\int_t^1 ud\left(\frac{1-\beta(u)}{u}\right)\right]\right|$$

(5.33)

$$=\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|\int_0^1 p_{nk}(t)d\left[\beta_2(t)-\int_t^1\frac{du}{u}\right]\right|$$

$$=\lim_{\lambda\to\infty}\sum_{k=m+1}^{n}\left|p_{nk}(1)(\beta(1)-\beta(1-0))-\int_0^1\frac{p_{nk}(t)}{t}dt\right|$$

(By Lemma (4.1) and the fact $p_{nk}(1) = 0$ for $0 \leqq k < n$)

$$= \lim_{\lambda \to \infty} \sum_{k=m+1}^{n-1} \frac{1}{\lambda_k} + \left| \beta(1) - \beta(1-0) - \frac{1}{\lambda_n} \right|$$

(by Lemma (5.6) since $q = 1$ and (2.1))

$$= \left| \beta(1) - \beta(1-0) \right|.$$

This proves our theorem for assumption (a) and $q = 1$, $m(\lambda) < n(\lambda)$ $(\lambda > \Lambda)$. The proof of our theorem for assumption (b) is similar. The proof of the second part of our theorem is similar, too. Q.E.D.

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